Discrete Applied Mathematics 43 (1993) 197–215
North-Holland

Recognizing a class of bicircular matroids

Collette R. Coullard*

School of Industrial Engineering and Management Sciences, Northwestern University, Evanston, IL 60208, USA

John G. del Greco

Department of Mathematical Sciences, Loyola University of Chicago, Chicago, IL 60626, USA

Donald K. Wagner**

Division of Mathematical Sciences, Office of Naval Research, Arlington, VA 22217, USA; and Department of Operations Research and Applied Statistics, George Mason University, Fairfax, VA 22030, USA

Received 16 November 1989
Revised 31 May 1991

Abstract

Coullard, C.R., J.G. del Greco and D.K. Wagner, Recognizing a class of bicircular matroids, Discrete Applied Mathematics 43 (1993) 197–215.

This paper presents a polynomial-time algorithm for solving a restricted version of the recognition problem for bicircular matroids. Given a matroid M, the problem is to determine whether M is bicircular. Chandru, Coullard and Wagner showed that this problem is NP-hard in general. The main tool in the development of the algorithm as well as the main theoretical contribution of the paper is a set of necessary and sufficient conditions for a given matroid to be the bicircular matroid of a given graph. As a final result, the complexity result of Chandru et al. is strenghtened.

Correspondence to: Professor J.G. del Greco, Department of Mathematics, Loyola University of Chicago, Chicago, IL 60626, USA.

* Research partially supported by Office of Naval Research Grant N00014-86-K-0689 at Purdue University and National Science Foundation Presidential Young Investigators Grant DDM-8957880 at Northwestern University.

** Research partially supported by Office of Naval Research Grant N00014-86-K-0689 at Purdue University.

0166-218X/93/$06.00

1. Introduction

This paper is concerned with a class of matroids defined on the edge set of a graph. To be precise, if $G=(V,E)$ is a graph and $\mathscr{I}$ is the set of those subsets A of E having the property that each component of $G[A]$ has at most one cycle, then $\mathscr{I}$ is a collection of independent sets of a matroid on E. This matroid is called the *bicircular* matroid of G and is denoted $B(G)$. A matroid M is called *bircircular* if there exists a graph G such that $M=B(G)$. In this case, G is called a *representation* of M. This paper addresses the *recognition problem* for bicircular matroids. Specifically, if M is a matroid, the recognition problem for M is to determine whether M is bicircular. Chandru, Coullard, and Wagner [2] proved that this problem is NP-hard, even when M is specified as the matric matroid of a given matrix.

This paper contains three main results on bicircular matroids. The first is structural. For a given matroid M and graph G, a set of necessary and sufficient conditions is obtained for M to be the bicircular matroid of G. This result is motivated in part by the fact that a set of similar conditions has been obtained by Seymour [13] characterizing when M is the polygon matroid [21] of G. Within the context of this paper, it serves as the basis for the second main result, a polynomial-time algorithm that solves a restricted version of the recognition problem for bicircular matroids. Specifically, the algorithm determines whether a given matroid is the bicircular matroid of a graph that is a member of a certain family of graphs. This family consists of the graphs homeomorphic from what are called here generalized wheels, pinwheels, and the graph K_5. (These graphs will be described in detail in the next section.) This family of graphs has arisen in the work of Coullard, del Greco, and Wagner [4] and Lovász [9]. Finally, the complexity result of Chandru et al. mentioned above is strengthened by proving that the recognition problem for bicircular matroids remains NP-hard for Halin graphs, a class of graphs closely related to generalized wheels.

The results of this paper are motivated by a desire to understand bicircular matroids, and in particular, to understand the similarities and differences between bicircular matroids and polygon matroids. Observe that the independent sets of the polygon matroid of a given graph G closely resemble those of the bicircular matroid of G, namely, a set A of edges of G is independent in the polygon matroid of G if and only if $G[A]$ is acyclic.

One difference between bicircular matroids and polygon matroids is revealed by contrasting their respective recognition problems. As mentioned above, the recognition problem for bicircular matroids is NP-hard, even when the given matroid is matric. However, Seymour [13] proved that the recognition problem for polygon matroids can be solved in polynomial time, even when the matroid is specified by an oracle. Seymour's algorithm uses the structural characterization of polygon matroids mentioned above together with any polynomial-time algorithm that determines whether a given binary matroid is a polygon matroid of a graph, and if so, constructs such a graph. (There exist many such algorithms.)

On the other hand, bicircular and polygon matroids both admit polynomial-time algorithms for the closely related *realization* problem. For a bicircular matroid M, the realization problem is to construct a graph representation of M. Indeed, when M is specified as the matric matroid of a given matrix over the set of real numbers, the algorithm of this paper coupled with that of Coullard et al. [4] yields a polynomial-time algorithm for the realization problem. A second such polynomial-time algorithm has been developed independently in a series of papers by Shull, Orlin, Shuchat, and Gardner [15] and Shull, Shuchat, Orlin, and Gardner [16]. The corresponding realization problem for polygon matroids is also solvable in polynomial time. In fact, essentially all of the algorithms for solving the recognition problem for polygon matroids also solve the realization problem since they determine whether the given matroid is the polygon matroid of some graph by attempting to construct one such graph. In this sense, the realization problem is a relaxation of the recognition problem.

The algorithms of Coullard et al. and Shull et al. for the realization problem for bicircular matroids were motivated to a large extent by certain connections that exists between bicircular matroids and linear programming. Both of these algorithms are aimed at the problem of converting a given linear-programming problem to an equivalent problem whose constraint matrix has special structure. Specifically, bicircular matroids are related to the matric matroids of the constraint matrices of a class of linear-programming problems called *generalized-network flow problems* (abbreviated gnf problems). The constraint matrices of these linear-programming problems, called *generalized-network flow* matrices (abbreviated gnf matrices), are characterized by having at most two nonzero entries per column. Gnf problems are important for at least two reasons. First, they have a number of applications, and second, they can be solved efficiently in practice [8]. Moreover, Goldberg, Plotkin, and Tardos [6] have developed a polynomial-time combinatorial algorithm for a subclass of gnf problems. A precise description of the relationship between bicircular matroids and gnf matrices is somewhat technical and is delayed until the next section. It suffices to say here that both algorithms exploit this relationship to solve a restricted version of the conversion problem mentioned above.

The remainder of the paper is outlined as follows. The next section introduces a number of preliminaries, including a description of how the results of this paper can be used to solve the realization problem for bicircular matroids. Section 3 contains the structural characterization of bicircular matroids, and Sections 4 and 5 contain the main algorithmic results. Finally, Section 6 contains the strengthening of the complexity result of Chandru et al.

2. Preliminaries

A general familiarity with graphs and matroids is assumed. For an introduction, see Bondy and Murty [1] and Welsh [21] respectively. For clarity, however, some definitions and notation are now established. Let G be a graph where V and E are the sets of vertices and edges of G respectively. The sets of vertices and edges of a graph G are also denoted by $V(G)$ and $E(G)$ respectively. If H is a graph, then $H \subseteq G$ denotes that H is a subgraph of G. If $S \subseteq E$, then $G[S]$ denotes the subgraph of G induced by S, and $G \setminus S$ denotes the subgraph obtained by deleting all the edges in S. In addition, G/S is the graph obtained from G by contracting the edges in S. Let $v \in V$. The graph $G \setminus \{v\}$ denotes the subgraph of G obtained by deleting v and all the edges of G having v as an end. Also, $\mathrm{st}_G(v)$, called the *star* of v, is the set of edges of G having v as an end. The vertex v is called *isolated* if $\mathrm{st}_G(v) = \emptyset$. Two subgraphs H_1 and H_2 of G are called *vertex disjoint* if $V(H_1) \cap V(H_2) = \emptyset$. A component of G is called a *tree* component if it contains no cycle and is called *cyclic* otherwise. If it contains exactly one cycle, then it is called a *1-tree*. The graph K_n denotes the complete graph on n vertices, and $K_{3,n}$, $n \geq 1$, denotes the complete bipartite graph with vertex partition $\{V_1, V_2\}$ where $|V_1| = 3$ and $|V_2| = n$. Two graphs G_1 and G_2 are *equal*, written $G_1 = G_2$, if one can be obtained from the other by renaming vertices. Finally, $\vec{G}$ denotes a digraph whose underlying graph is G.

A matrix A is called a *vertex-edge incidence* matrix of a graph G if the rows and columns of A are indexed over $V(G)$ and $E(G)$ respectively, and the (v, e) entry of A, $v \in V(G)$, $e \in E(G)$, is a 1 if and only if v is an end of e. Two matrices of the same dimensions are said to have the same nonzero pattern if corresponding entries are either both zero or nonzero.

Let $M := (E, \Phi)$ be a matroid where E is the ground set and Φ is the set of circuits of M. For $S \subseteq E$, $M \setminus S$ and M/S denote the matroids obtained from M by deleting and contracting the elements in S respectively. Two elements e and f are in *series* if $\{e, f\}$ is a cocircuit. A *series class* of M is a maximal subset of E every pair of elements of which is in series. The matroid M^* is the dual of M. If C is a circuit and D is a cocircuit, then $|C \cap D| \neq 1$. This property is called *orthogonality*. If B is a base of M and $e \in E - B$, then $B \cup \{e\}$ contains a unique circuit called the *fundamental circuit* of $B \cup \{e\}$. If r is the rank function of M, then M is called *connected* if there does not exist a partition $\{E_1, E_2\}$ of E such that $|E_1| \geq 1 \leq |E_2|$ and $r(E_1) + r(E_2) = r(E)$. A *component* of M is a maximal set $S \subseteq E$ such that $M \setminus (E - S)$ is connected. The *rank* of M, that is, the value of $r(E)$, will be denoted $\varrho(M)$. Matroids $M_1 := (E_1, \Phi_1)$ and $M_2 := (E_2, \Phi_2)$ are *equal*, written $M_1 = M_2$, if $E_1 = E_2$ and $\Phi_1 = \Phi_2$. Finally, if A is a matrix, then $M(A)$ denotes the matric matroid of A, and A is called a *representation* of $M(A)$.

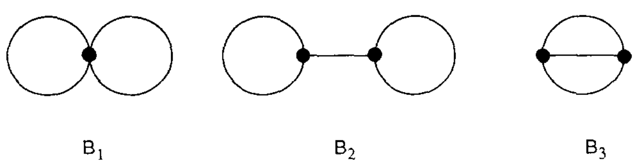

Fig. 1.

A graph is called a *bicycle* if it is homeomorphic from one of the graphs B_1, B_2, or B_3 in Fig. 1. If a graph is homeomorphic from B_i, $1 \le i \le 3$, then it is called a *type-i* bicycle. If G is a graph, the edge sets of the bicycles of G are the circuits of $\boldsymbol{B}(G)$. A bicircular matroid may have more than one representation. For example, the graphs B_2 and B_3 in Fig. 1 represent the same matroid assuming $E(B_2)=E(B_3)$. Bicircular matroids were introduced by Simões-Pereira [17, 18] and later studied by Matthews [10, 11] and others [2, 3, 12, 14, 20, 22–24]. If $S \subseteq E$, then the rank of S in $\boldsymbol{B}(G)$ is $|V(G[S])| - |\Upsilon(G[S])|$, where $\Upsilon(G[S])$ is the set of tree components of $G[S]$. Furthermore, S is a cocircuit of $\boldsymbol{B}(G)$ if and only if it is a minimal set of edges of G whose deletion increases the number of tree components. In addition, if G has at least two edges, $\boldsymbol{B}(G)$ is connected if and only if G has no degree-one vertices and is not a cycle [10].

As stated in Section 1, bicircular matroids are related to the matric matroids of gnf matrices. To make this relationship precise, it is convenient to introduce a class of matroids that contains the bicircular matroids as a proper subclass. This class of matroids will be referred to again in Section 6.

The pair $(\vec{G}, w)$ is called a *weighted digraph* if $w: E \to \mathbb{R} - \{0\}$, where G is a graph having edge set E and $\mathbb{R}$ is the set of real numbers. The function w specifies the *weights* on the edges of G. An *oriented* cycle of a graph is a cycle to which one of the two possible orientations has been assigned.

Let $(\vec{G}, w)$ be a weighted digraph where G has edge set E. If C is an oriented cycle of G, R is the set of edges of C that are directed in $\vec{G}$ opposite to the orientation of C (reverse edges), and F is the set of the remaining edges of C (forward edges), then C is called a *unit-gain* cycle of $(\vec{G}, w)$ if

$$\left(\prod_{e \in F} w(e)\right)\left(\prod_{e \in R} w(e)^{-1}\right) = 1.$$

The set of edge sets of the unit-gain cycles of $(\vec{G}, w)$ along with those of the bicycles of G no cycle of which is unit-gain is the set of circuits of a matroid on E called the *gain* matroid of $(\vec{G}, w)$ and is denoted $\mathbf{GN}(\vec{G}, w)$ [22]. Note that if $(\vec{G}, w)$ has no unit-gain cycles, then $\mathbf{GN}(\vec{G}, w) = \boldsymbol{B}(G)$. A matroid M is called a *gain* matroid if there exists a weighted digraph $(\vec{G}, w)$ such that $M = \mathbf{GN}(\vec{G}, w)$. Bicircular matroids are gain matroids. To see this, let G be a graph with edge set E, and let $w: E \to P$ where P is the set of prime numbers and w is injective. Clearly, $\mathbf{GN}(\vec{G}, w) = \boldsymbol{B}(G)$.

A gain matroid is matric and moreover has a representation which is a gnf matrix. To see this, consider the gain matroid $\mathbf{GN}(\vec{G}, w)$, and let N be a matrix whose rows and columns are indexed over $V(G)$ and $E(G)$ respectively and whose entries are defined as follows: if $v \in V(G)$ and $e \in E(G)$, then the (v, e) entry of N is

$$n_{(v,e)} = \begin{cases} -1, & \text{if } e \text{ is directed out of } v, \\ w(e), & \text{if } e \text{ is directed into } v \text{ and } e \text{ is not a loop, and} \\ 0, & \text{otherwise.} \end{cases}$$

Using standard linear-algebra techniques, it can be shown that the set of bases of N coincides with the set of bases of $\mathbf{GN}(\vec{G}, w)$ [8]. Consequently, $M(N) = \mathbf{GN}(\vec{G}, w)$. Call N the *associated matrix* of $(\vec{G}, w)$. The preceding construction can be reversed. Specifically, if N is a gnf matrix, let N' be a matrix obtained from N by scaling the columns of N so that each column has a -1 entry. (Note that $M(N) = M(N')$.) Clearly, there exists a weighted digraph, called an *associated weighted digraph* of N, whose associated matrix is N'. To summarize, gain matroids are the matric matroids of gnf matrices, and moreover, bicircular matroids are the matric matroids of gnf matrices whose associated weighted digraphs have no unit-gain cycles. Such matrices are called *bicircular* gnf matrices. (Note that if N is a gnf matrix, then any two associated weighted digraphs of N have the same set of unit-gain cycles.)

The remainder of this section describes how to use the algorithm of Coullard et al. [4] together with the results of this paper to solve the realization problem for bicircular matroids.

Let A be a full-row-rank matrix. The matrix A is said to have *property* $\boldsymbol{B}$ if it is row equivalent to a bicircular gnf matrix. (Clearly, if A has property $\boldsymbol{B}$, then $M(A)$ is bicircular.) The algorithm of Coullard et al. mentioned above is a polynomial-time algorithm, called algorithm TRANSFORM, for solving the realization problem for the matroid $M(A)$ assuming $M(A)$ is connected and A has property $\boldsymbol{B}$. The following proposition addresses the case where $M(A)$ is connected and bicircular but does not have property $\boldsymbol{B}$.

Proposition 2.1. *Let A be a full-row-rank matrix. If $M(A)$ is connected and bicircular but does not have property $\boldsymbol{B}$, then for any representation G of $M(A)$ that has no isolated vertices, the following hold:*

(i) *every vertex has degree at least two,*

(ii) *every pair of cycles share a vertex, and*

(iii) *no vertex is contained in every cycle.*

Proof. Let G be a representation of $M(A)$. Suppose first that G has a degree-one vertex. Since $\boldsymbol{B}(G)$ is connected, G consists of a single edge. Therefore, A is 1×1 and its single entry is nonzero. Clearly then, A has property $\boldsymbol{B}$, a contradiction. Therefore, (i) holds.

Now suppose that G has either a vertex that is contained in every cycle or a pair of vertex-disjoint cycles. Now the matroid $M(A)$ has a representation G' such that the star of every vertex of G' is a cocircuit of $\boldsymbol{B}(G')$ [3, Lemma 4.4]. Moreover, if the graph G has a vertex that is contained in every cycle, G' has a loop. In addition, if G has a pair of vertex-disjoint cycles, then so does G'. Since the star of every vertex of G' is a cocircuit of $\boldsymbol{B}(G')$, there exists a matrix T such that TA has the same nonzero pattern as a vertex-edge incidence matrix of G' [3, Proposition 5.1]. Since G' has either a loop or a pair of vertex-disjoint cycles, this matrix T is nonsingular [3, Proposition 5.2]. Therefore, A has property $\boldsymbol{B}$, a contradiction. This proves (ii) and (iii). □

An algorithm for the realization problem for a connected bicircular matroid $M(A)$ can now be described assuming A has full row rank. First, use the algorithms of Sections 4 and 5 to test if $M(A)$ has a representation satisfying conditions (i)–(iii) of Proposition 2.1, and if so, produce such a representation. If not, then $M(A)$ has property $\boldsymbol{B}$. In this case, apply algorithm TRANSFORM to obtain a representation of $M(A)$.

The assumption that $M(A)$ is connected is made without loss of generality. To see why, suppose that A is $m \times n$, and assume that the first m columns of A constitute an identity matrix. An $m \times n$ matrix having this property is in *standard form*. (Since A has full row rank, A can be converted into a matrix in standard form in polynomial time.) Suppose that the rows and columns of A are indexed over the sets R and C respectively. Construct a bipartite graph G with vertex partition $\{R, C\}$ and having an edge with ends $r \in R$ and $c \in C$ if and only if the (r, c) entry of A is nonzero. The graph G is connected if and only if $M(A)$ is connected [19]. It follows that if G has k components, then the rows and columns of A can be permuted so that A can be written in block diagonal form as

$$\begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_k \end{bmatrix}$$

where $M(A_i)$, $1 \le i \le k$, is a component of $M(A)$. Note that for $1 \le i \le k$, A_i is in standard form. Since deletion minors of bicircular matroids are bicircular [10], algorithm TRANSFORM and the algorithms of this paper can be used to construct for $1 \le i \le k$ a graph G_i such that $M(A_i) = \boldsymbol{B}(G_i)$. Clearly, $M(A)$ is the bicircular matroid of the graph whose components are the graphs G_i, $1 \le i \le k$. Finally, it should be noted that the matrices A_i, $1 \le i \le k$, can be computed in $O(n^2)$ time [7].

The last result of this section is a characterization of the graphs that satisfy conditions (i)–(iii) of Proposition 2.1.

A connected loopless graph G having at least four vertices is called a *generalized wheel* if there exists a vertex v, called a *hub* vertex, such that $G \setminus \{v\}$ is a cycle every vertex of which is adjacent to v. A vertex of G that is not v is called a *rim* vertex. If an edge of G has v as an end, then it is called a *spoke*. Otherwise, it is called a *rim* edge. A maximal set of spokes every pair of which has the same set of ends is called a *spoke class*. If every spoke class has only one spoke, then G is called a *wheel*. Note that if G has at least five vertices, then G has a unique hub vertex.

A connected loopless graph with vertex set R such that $|R| = 3$ is called a *pinwheel* if every pair of vertices of R are joined by at least two edges. A connected loopless graph G with at least four vertices is called a *pinwheel* if there exists a subset R of $V(G)$ such that $|R| = 3$, every vertex in R has degree at least three, and every vertex not in R is joined to every vertex in R by exactly one edge. The vertices in R are called *rim* vertices, and the edges having both ends in R are called *rim* edges. An edge that is not a rim edge is called a *spoke*. The subgraph of G induced by the three spokes incident to a nonrim vertex is called a *wing*. Note that the subgraph induced by the set of spokes is $K_{3,n}$ for some $n \geq 1$.

A version of the following theorem is due to Lovász [9].

Theorem 2.2. *A graph satisfies conditions* (i)–(iii) *of Proposition* 2.1 *if and only if it is homeomorphic from* K_5, *a generalized wheel, or a pinwheel.*

The next three sections are devoted to developing polynomial-time algorithms for testing whether a matroid M is a bicircular matroid with a representation that satisfies conditions (i)–(iii). The algorithms do not assume that M is bicircular.

3. Characterizing bicircular matroids

The purpose of this section is to identify necessary and sufficient conditions for a given matroid to be the bicircular matroid of a given graph.

Theorem 3.1. *If M is a matroid and G is a connected graph, then $M = B(G)$ if and only if*

(i) *the star of every vertex of G is a union of cocircuits of M,*
(ii) *the edge set of a vertex-disjoint union of cycles of G is independent in M, and*
(iii) $\varrho(M) \leq \varrho(B(G))$.

Proof. ($\Rightarrow$) Conditions (ii) and (iii) hold trivially. If v is a vertex of G, then since $G \setminus \mathrm{st}_G(v)$ has at least one more tree component than does G (the isolated vertex v, for example), $\mathrm{st}_G(v)$ contains a cocircuit D_1 of $B(G)$. If $D_1 = \mathrm{st}_G(v)$, the result follows. Otherwise, let $G' := G \setminus D_1$. By the same argument, $\mathrm{st}_{G'}(v)$ contains a cocircuit D_2 of $B(G')$. However, $B(G') = B(G) \setminus D_1$ [10], and so $D_2 \cup S$ is a cocircuit of $B(G)$ where $S \subseteq D_1 \subseteq \mathrm{st}_G(v)$. If $D_1 \cup D_2 = \mathrm{st}_G(v)$, then, as before, the result is proved. Otherwise, let $G'' := G' \setminus D_2$. Continuing this process yields the result.

($\Leftarrow$) **Claim.** *If $H \subseteq G$ and each component of H is either a tree or a* 1*-tree, then $E(H)$ is independent in M.*

Proof of Claim. Suppose the claim is false, and let H be a counterexample with $|E(H)|$ a minimum. Note that $E(H) \neq \emptyset$. If H is a union of cycles, then $E(H)$ by assumption is independent in M, a contradiction. Therefore, H has a degree-one vertex v. Let e be the edge of H that has v as an end. By condition (i), e is an element of a cocircuit D of M such that $D \cap E(H) = \{e\}$. Therefore, if C is a circuit of M contained in $E(H)$, then $e \notin C$ by orthogonality. Since each component of $H \setminus \{e\}$ is a tree or a 1-tree and $H \setminus \{e\}$ has one less edge than H, $E(H \setminus \{e\})$ is independent in M, a contradiction, since by assumption, $E(H \setminus \{e\})$ contains a circuit of M.

Now let B be a base of $B(G)$. Since $G[B]$ is the vertex-disjoint union of trees and 1-trees, B is independent in M by the claim. Therefore, $\varrho(B(G)) \leq \varrho(M)$, and so by (iii), M and $\boldsymbol{B}(G)$ have the same rank. Therefore, B is a base of M. It remains to show that every base of M is a base of $\boldsymbol{B}(G)$. Let B be a base of M, and suppose B is dependent in $\boldsymbol{B}(G)$. In this case, $G[B]$ contains a bicycle K of G. If K is a type-2 bicycle, let e be an edge of one of the cycles of K. Otherwise, choose e arbitrarily. Since $E(K \setminus \{e\})$ is independent in $\boldsymbol{B}(G)$ and G is connected, $E(K \setminus \{e\})$ can be extended to a base B' of $\boldsymbol{B}(G)$ such that $G[B']$ is connected. Since M and $\boldsymbol{B}(G)$ have the same rank, B' is a base of M. Therefore, $B' \cup \{e\}$ contains a unique circuit C of M. By orthogonality, $G[C]$ has no degree-one vertices. By condition (ii), $E(K) = C$, a contradiction, since $C = E(K) \subseteq B$. So B is independent in $\boldsymbol{B}(G)$, and again, since M and $\boldsymbol{B}(G)$ have the same rank, B is a base of $\boldsymbol{B}(G)$ completing the proof. $\square$

If condition (ii) is deleted and (iii) is replaced by

(iii′) $\varrho(M) \leq \varrho(\boldsymbol{P}(G))$

where $\boldsymbol{P}(G)$ is the polygon matroid of G, then Seymour proved that $M = \boldsymbol{P}(G)$ if and only if conditions (i) and (iii′) hold [13].

Condition (ii) is indispensable. For example, let G be a type-2 bicycle both cycles of which have two edges, and let $M := \boldsymbol{P}(G)$. Conditions (i) and (iii) of Theorem 3.1 hold, but $M \neq \boldsymbol{B}(G)$. The following special case will be needed later.

Corollary 3.2. *If M is a matroid and G is a graph that has no pair of vertex-disjoint cycles, then $M = \boldsymbol{B}(G)$ if and only if*

(i) *the star of every vertex of G is a union of cocircuits of M,*

(ii) *the edge set of every cycle of G is independent in M, and*

(iii) $\varrho(M) \leq \varrho(\boldsymbol{B}(G))$.

4. Generalized wheels

In this section, a polynomial-time algorithm is described that determines whether an arbitrary matroid is the bicircular matroid of a graph that is homeomorphic from a generalized wheel. If so, the algorithm constructs an appropriate graph. Otherwise, the algorithm returns the conclusion that no such graph exists. The validity of the algorithm along with that of the algorithm of the next section that recognizes bicircular matroids of graphs homeomorphic from pinwheels will follow easily from Corollary 3.2.

Let Γ_w be the subclass of generalized wheels having at least four spoke classes.

Proposition 4.1. *If G is a wheel in Γ_w, then a three-element cocircuit of $\boldsymbol{B}(G)$ is the star of a vertex.*

Proof. Let v be the hub vertex of G, and let D be a three-element cocircuit of $\boldsymbol{B}(G)$. Suppose that there are at least k spokes remaining after the spokes in D are deleted from G. Since G has at least four spokes, $k \geq 1$. If $k \geq 4$, then v is clearly a vertex of a cyclic component of $G \setminus D$. If $1 \le k \le 3$, then D contains exactly $k-1$ rim edges, and so again, v is a vertex of a cyclic component of $G \setminus D$. Let T be the unique tree component of $G \setminus D$. Suppose T has a degree-one vertex. Therefore, T has at least two such vertices u and w. Since v is not a vertex of T, there are edges in D with ends u and v, and w and v respectively. The remaining edge in D has ends u and w. In this case, G has only two spokes, a contradiction. Therefore, T is an isolated vertex, and so D is the star of a vertex, completing the proof. □

Clearly, if G is a generalized wheel with at least three spoke classes, then the star of every vertex of G is a cocircuit of $B(G)$.

Let M be a matroid with ground set E, $|E| \ge 8$. Algorithm WR outlined below (Wheel Recognition) either returns a wheel in Γ_w whose bicircular matroid is M or terminates with the conclusion that no such wheel exists. (Generalized wheels will be treated later.) A discussion of how each task required by the algorithm might be carried out along with an analysis of the complexity of the algorithm follows the outline. Text enclosed within "(*" and "*)" is a comment. Also, the instruction "**STOP**" means that M is not the bicircular matroid of a wheel in Γ_w.

Algorithm WR.

Input: A matroid M specified by a ground set E, $|E| \ge 8$, and an independence oracle.

Output: A wheel G in Γ_w such that $M = B(G)$ or the conclusion that no such wheel exists.

Step WR1 (* Construct a wheel G *). Determine whether the elements of E can be labeled and partitioned into two sets $S := \{e_i \mid 1 \le i \le n\}$ and $R := \{f_i \mid 1 \le i \le n\}$, such that the sequence of sets $\langle \{f_1, e_1, f_2\}, \{f_2, e_2, f_3\}, \ldots, \{f_{n-1}, e_{n-1}, f_n\}, \{f_n, e_n, f_1\} \rangle$ represents an ordering of all the three-element cocircuits of M. (* Note that $n \ge 4$. *) If the sets S and R do not exist, **STOP**. Otherwise, let G be a wheel such that S is the set of spokes, R is the set of rim edges, and for $1 \le i \le n$, f_i is adjacent to e_i and e_{i+1} (with subscripts taken mod n).

Step WR2 (* Check if $M = B(G)$ *). If $\varrho(M) > |V(G)|$, **STOP**. (* Note that $|V(G)| = \varrho(B(G))$ *) If S is not a cocircuit of M, **STOP**. If there is a cycle of G whose edge set is dependent in M, **STOP**. Otherwise, output G and **END**.

Step WR1 of Algorithm WR can be accomplished as follows. First, generate the set Δ of all three-element cocircuits of M. For each $D \in \Delta$, determine whether D has a unique element that is contained in no other member of Δ. If such an element does not exist for some member of Δ, Step WR1 cannot be completed. Otherwise, choose some member of Δ and call it D_1. Let e_1 denote the unique element of D_1 that is contained in no other member of Δ. Label the other two elements of D_1 as f_1 and f_2. In general, suppose $f_1, e_1, f_2, e_2, f_3, \ldots, f_k, e_k$, and f_{k+1} have been labeled, $k \ge 1$. If there does not exist exactly one other member of Δ (excluding D_k) containing f_{k+1}, then again Step WR1 cannot be completed. Otherwise, let D_{k+1} denote this cocircuit, and let e_{k+1} denote the unique element of D_{k+1} that is contained in no other member of Δ. Assuming that it has not already been labeled, label the remaining unlabeled element of D_{k+1} as f_{k+2}. Continuing this process, there will come a point at which some three-element cocircuit $D_{k'}$ is chosen containing the previously labeled element $f_{k'}$. After labeling the unique element of $D_{k'}$ that is not contained in any other member of Δ as $e_{k'}$, the remaining element of $D_{k'}$ will have already been labeled in a previous stage. Call this element f. There are two cases to consider. Suppose first that $f \neq f_1$. In this case, Step WR1 cannot be completed. Now suppose $f = f_1$. In this case, if the sequence $\langle f_1, e_1, f_2, e_2, f_3, \ldots, f_{k'}, e_{k'}, f_1 \rangle$ contains all the elements of E, then the sets S and R exist. Otherwise, Step WR1 cannot be completed.

The total amount of work is dominated by the time required to compute the set Δ. Using the independence oracle, Δ can be computed in $O(|E|^4)$ time. (If r^* is the corank function of M, then for $X \subseteq E$, $r^*(X) = |X| - r(E) + r(E-X)$ [21]. Consequently, $r^*(X)$ can be computed in $O(|E|)$ time using the greedy algorithm, and so testing if X is a cocircuit requires $O(|X|\,|E|)$ time.) It takes $O(|E|)$ time to compute $\varrho(M)$ and $O(|E|^2)$ time to determine if S is a cocircuit of M. Finally, generating and testing the edge sets of the cycles for independence can be done in $O(|E|^3)$ time.

Suppose $M = \boldsymbol{B}(G)$ where G is a wheel in Γ_w. By Proposition 4.1, the set of three-element cocircuits of M is precisely the set of stars of the rim vertices of G. Since an edge of G is a rim edge if and only if it is an element of the stars of exactly two rim vertices, the graph constructed in Step WR1 of Algorithm WR is the graph G. If, however, M is not the bicircular matroid of such a wheel, then the algorithm will conclude this by Corollary 3.2.

The preceding observations are summarized in the following theorem.

Theorem 4.2. *Algorithm* WR *is correct and requires* $O(|E|^4)$ *time.*

Suppose $M = \boldsymbol{B}(G)$ where G is a generalized wheel in Γ_w that has a spoke class with at least two edges. Assume that a mechanism can be devised to recognize when two elements of M are in the same spoke class of G. In this case, the spoke classes of G can be determined. Suppose $S_1, S_2, \ldots, S_r$ are the spoke classes, and for $1 \leq i \leq r$, choose $e_i \in S_i$. Let

$$S := \bigcup_{i=1}^{r} (S_i - \{e_i\}).$$

Since $M = \boldsymbol{B}(G)$,

$$M \setminus S = \boldsymbol{B}(G) \setminus S = \boldsymbol{B}(G \setminus S).$$

Clearly, $G \setminus S$ is a wheel. Therefore, Algorithm WR will return the graph $G \setminus S$ when it is applied to $M \setminus S$. The graph G can now be recovered by adding the elements in S to $G \setminus S$ as spokes so that for $1 \leq i \leq r$, the spoke class containing e_i is S_i. The

(a) (b)

Fig. 2.

next proposition provides a way to determine when two elements of M are in the same spoke class of G.

Proposition 4.3. *If G is a generalized wheel in Γ_w all of whose spoke classes have at most two edges, then edges e and f of G are in the same spoke class if and only if there exists two four-element circuits C_1 and C_2 of $B(G)$ such that $C_1 \cap C_2 = \{e, f\}$.*

Proof. Call a four-edge bicycle of G *type* A if it is isomorphic to the graph of Fig. 2(a) and *type* B if it is isomorphic to the graph of Fig. 2(b). Clearly, every four-edge bicycle of G is either type A or type B. Suppose v is the hub vertex of G.

($\Rightarrow$) Suppose the edges e and f are in the same spoke class of G. Let a and b be the rim edges of G that are incident to both e and f, and let c and d be spokes neither of which is e or f, and that are incident to a and b, respectively. Now let $C_1 := \{a, c, e, f\}$ and $C_2 := \{b, d, e, f\}$. Clearly, $G[C_1]$ and $G[C_2]$ are type-A bicycles.

($\Leftarrow$) Let $B_1 := G[C_1]$ and $B_2 := G[C_2]$. First suppose that both B_1 and B_2 are type-A bicycles. Suppose B_1 and B_2 share a rim edge a. If u and w are the ends of a, then each of the five edges in $(C_1 \cup C_2) - \{a\}$ either has ends u and v or w and v. By the pigeon-hole principle, G therefore has a spoke class with at least three edges, a contradiction. So e and f are both spokes. If e and f do not have the same ends, then B_1 and B_2 must also share a rim edge, a contradiction.

Now suppose that B_1 is type A and B_2 is type B. Note that e and f are both spokes in this case. If e and f do not have the same ends, and u and w are the ends of the rim edge to which e and f are both incident, then, as before, each of the remaining five edges in $C_1 \cup C_2$ either has ends u and v, or w and v, a contradiction.

Finally, suppose that both B_1 and B_2 are type B. If e and f do not have the same ends and u and w are the rim vertices of G that are ends of e and f respectively, then each of the four edges in $(C_1 \cup C_2) - \{e, f\}$ either has ends u and v, or w and v, a contradiction. $\square$

If $M = B(G)$ where G is a generalized wheel in Γ_w, the spoke classes of G can be easily identified by generating the three-element circuits of M and then if necessary generating the four-element circuits of M and applying Proposition 4.3. (Note that C is a three-element circuit of $B(G)$ if and only if $G[C]$ is a three-edge type-3 bicycle. Therefore, the elements of C are in the same spoke class.) An algorithm to determine whether a matroid is the bicircular matroid of a generalized wheel in Γ_w runs generally as in the description prior to the proposition. The details are left to the reader. The algorithm still requires $O(|E|^4)$ time.

Finally, consider the problem of recognizing when a matroid M with ground set E is the bicircular matroid of a graph that is homeomorphic from a generalized wheel in Γ_w. Call a subgraph of a graph a *line* if it is a path whose internal vertices (if any) have degree two and whose ends have degree at least three. Suppose that $M = \boldsymbol{B}(G)$ where G is homeomorphic from a generalized wheel in Γ_w. Consider any graph obtained from G by contracting all but one edge from every line of G. If S is the set of edges contracted, then [10]

$$M/S = \boldsymbol{B}(G)/S = \boldsymbol{B}(G/S).$$

Clearly, G/S is a generalized wheel in Γ_w, and consequently, Algorithm WR can be used to construct it. A mechanism is therefore needed to recognize when an element of M is an edge of a line of G that has at least two edges. If $S \subsetneq E$, S is a series class of M if and only if $G[S]$ is a line [3]. To *cosimplify* a matroid means to contract all but one element from every series class. To obtain an algorithm to recognize when a matroid is the bicircular matroid of a graph that is homeomorphic from a generalized wheel in Γ_w, it is therefore only necessary to first cosimplify the matroid and then to use Algorithm WR. Suppose $\bar{M} = \boldsymbol{B}(\bar{G})$ where $\bar{M}$ denotes a cosimplification of M and $\bar{G}$ is the generalized wheel obtained by using Algorithm WR. If $\bar{E}$ is the ground set of $\bar{M}$ and $e \in \bar{E}$, let L_e denote the series class of M containing e. Let $\hat{G}$ be the graph obtained from G by adding the elements of

$$\bigcup_{e \in \bar{E}} (L_e - \{e\})$$

to $\bar{G}$ as edges so that for $e \in \bar{E}$, $\hat{G}[L_e]$ is a line. Clearly, $\boldsymbol{B}(\hat{G}) = \boldsymbol{B}(G)$ since permuting the edges within a line of a graph preserves the bicircular matroid of the graph [3]. Since cosimplifying M requires only $O(|E|^3)$ time, the recognition algorithm just described needs $O(|E|^4)$ time. A detailed description of the algorithm is left to the reader.

Recognizing when a matroid is the bicircular matroid of a graph that is homeomorphic from a generalized wheel not in Γ_w can be easily done by cosimplifying, identifying the spoke classes, and enumerating. Again, the details are left to the reader.

5. Pinwheels

In this section, polynomial-time algorithms are described that determine whether an arbitrary matroid is the bicircular matroid of a graph that is homeomorphic from a pinwheel or a K_5. If so, the algorithm constructs an appropriate graph. Otherwise, the algorithm returns the conclusion that no such graph exists.

Let Γ_p denote the subclass of pinwheels having at least three wings.

Proposition 5.1. *If G is a pinwheel in Γ_p, then a three-element cocircuit of $B(G)$ is the star of a vertex of G.*

Proof. Let D be a three-element cocircuit of $B(G)$, and let T be the unique tree component of $G \setminus D$. Suppose T has a degree-one vertex. Therefore, T has at least two such vertices u and v. Since G has no degree-one or degree-two vertices and $|D|=3$, there exists an $e \in D$ that has ends u and v. If T has a third degree-one vertex, then it has exactly three degree-one vertices. In this case, G is isomorphic to K_4, a contradiction. Therefore, T is a path. If T has a degree-two vertex, then since every vertex of G has degree at least three, at least two edges of D have both their ends in T. Therefore, T is the only component of $G \setminus D$ for otherwise G has a cut vertex or is disconnected. It follows that G has at most six edges, a contradiction. Consequently, T has exactly one edge, and so since no two spokes have the same ends, e is a rim edge of G, and u and v are rim vertices. Note that both u and v have degree at most four. However, since u and v are rim vertices and G has at least three wings, both have degree at least five, a contradiction. Therefore, T is an isolated vertex, and D is the star of this vertex. □

Clearly, if G is in Γ_p, the star of every vertex of G is a cocircuit of $B(G)$.

Let M be a matroid with ground set E, $|E| \geq 9$. Algorithm PR outlined below (Pinwheel Recognition) either returns a pinwheel in Γ_p whose bicircular matroid is M or terminates with the conclusion that no such pinwheel exists. The algorithm proceeds roughly as follows. In Step PR1, three disjoint three-element cocircuits of M are selected (if possible) in order to construct a $K_{3,3}$ which must be a subgraph of the pinwheel in Γ_p, if it exists. If $|E|>9$, then in Step PR2, those vertices of the $K_{3,3}$ constructed in the previous step that will be the rim vertices of the pinwheel are identified. In Steps PR3 and PR4, the wings and rim edges of the pinwheel are determined as well as the way they should be attached to the $K_{3,3}$ output in Step PR1. In Step PR5, the wings and rim edges are assembled to produce the pinwheel. Finally, in Step PR6, it is decided whether the bicircular matroid of the pinwheel constructed in the previous step is the matroid M.

Algorithm PR.

Input: A matroid M specified by a ground set E, $|E| \geq 9$, and an independence oracle.

Output: A pinwheel G in Γ_p such that $M = B(G)$ or the conclusion that no such pinwheel exists.

Step PR1 (* Construct an initial $K_{3,3}$ *). If M does not have three pairwise disjoint three-element cocircuits, **STOP**. Otherwise, choose three such cocircuits D_1, D_2, and D_3, and let $S := D_1 \cup D_2 \cup D_3$. Let Γ denote the set of graphs that are isomorphic to $K_{3,3}$ and have D_1, D_2, and D_3 as stars. If there does not exist a graph in Γ the set of whose stars is the set of three-element cocircuits of $M \setminus (E-S)$, **STOP**. Otherwise, let K denote this $K_{3,3}$. (* Note that if K exists and M is the bicircular matroid of a pinwheel in Γ_p, then K is unique. $*$) If $E=S$, let $G:=K$, $R:=\emptyset$, and **GO TO** Step PR6. Otherwise, let U be the set of three vertices of K whose stars are D_1, D_2, and D_3, and let $W:=V(K)-U$. ($*$The sets U and W are the candidates for the set of rim vertices of the pinwheel to be constructed. $*$)

Step PR2 ($*$Identify the rim vertices $*$). Since $E\neq S$, choose an $e\in E-S$. Suppose first that e is an element of some three-element cocircuit D of M. If D is not unique or is not disjoint from S, **STOP**. Let Γ be the set of pinwheels with four wings that can be obtained from K by adding the elements of D to K as edges so that for $G\in\Gamma$, $G[D]$ is a wing and the set of rim vertices of G is either U or W.

Now suppose that e is in no three-element cocircuit of M. In this case, let Γ be the set of pinwheels with three wings that can be obtained from K by adding e to K as an edge so that e has both its ends in either U or W.

In either case, if there does not exist a unique graph in Γ whose four-edge stars are cocircuits of $M\backslash(E-(S\cup D))$ or $M\backslash(E-(S\cup\{e\}))$ respectively, **STOP**. Otherwise, such a graph exists, and in this case, let v be any one of its degree-four vertices. If v is a vertex in U, let $R:=U$. Otherwise, let $R:=W$. ($*$The set R is the set of rim vertices of the pinwheel to be constructed. $*$)

Step PR3 ($*$Identify the wings $*$). Let Δ be the set of three-element cocircuits that are disjoint from S. ($*$If M is the bicircular matroid of a pinwheel in Γ_p and there is a three-element cocircuit not disjoint from S, then there is at most one, and it is the star of a vertex in R. $*$) If $\Delta=\emptyset$, **GO TO** Step PR4. If the cocircuits in Δ are not pairwise disjoint, **STOP**. Otherwise, for each $D\in\Delta$, do the following. Let Γ be the set of pinwheels with four wings that can be obtained from K by adding the elements of D to K as edges so that for $G\in\Gamma$, $G[D]$ is a wing and the set of rim vertices of G is R. If there does not exist a unique graph in Γ whose four-edge stars are cocircuits of $M\backslash(E-(S\cup D))$, **STOP**. Otherwise, let G_D denote this unique graph.

Step PR4 ($*$Identify the rim edges $*$). Let T be the union of the cocircuits in Δ. (If $\Delta=\emptyset$, let $T:=\emptyset$.) If $S\cup T=E$, then **GO TO** Step PR5. Otherwise, for each $e\in E-(S\cup T)$ do the following. Let Γ be the set of graphs that can be obtained from K by adding e to K as an edge with ends in R. If there does not exist a unique graph in Γ whose two four-edge stars are cocircuits of $M\backslash(E-(S\cup\{e\})$, **STOP**. Otherwise, let G_e denote this unique graph.

Step PR5 ($*$Construct a pinwheel $G*$). Let G denote the pinwheel with $|\Delta|+3$ wings such that for every $D\in\Delta$ and $e\in E-(S\cup T)$, G_D and G_e are subgraphs of G.

Step PR6 ($*$Check if $M=B(G)*$). If $\varrho(M)>|V(G)|$, **STOP**. If the stars of the vertices in R are not cocircuits of M, **STOP**. If there is a cycle of G whose edge set is dependent in M, **STOP**. Otherwise, output G and **END**.

The time needed by Algorithm PR is dominated by the time required to generate the three-element cocircuits of M and test whether the edge set of every cycle of G is independent in M. Both operations can be done in $O(|E|^4)$ time.

Suppose that $M=B(G)$ where G is a pinwheel in Γ_p. If G is isomorphic to $K_{3,3}$, then it follows from Proposition 5.1 that Step PR1 will construct G. The validity of Step PR2 follows again from Proposition 5.1 and from the fact that if there is a graph in Γ whose four-edge stars are cocircuits of $M\backslash(E-(S\cup D))$ (or $M\backslash(E-(S\cup\{e\}))$), then those cocircuits are not cocircuits of the bicircular matroid of any other graph in Γ. (This can be easily verified by case analysis.) The validity of Steps PR3 and PR4 follow from similar arguments. Therefore, the graph G is the one constructed in Step PR5. If, on the other hand, M is not the bicircular matroid of a pinwheel in Γ_p, the algorithm will conclude this by Corollary 3.2.

Theorem 5.2. *Algorithm* PR *is correct and requires* $O(|E|^4)$ *time.*

As is the case with generalized wheels, there exists an algorithm to determine whether a matroid is the bicircular matroid of a graph that is homeomorphic from a pinwheel in Γ_p. Also, enumeration-based procedures can be easily constructed to recognize bicircular matroids of pinwheels not in Γ_p. In both cases, the details are left to the reader.

To determine whether a matroid is the bicircular matroid of a graph homeomorphic from K_5, cosimplify and then enumerate.

6. A negative result for Halin graphs

A connected planar graph G with edge set $T\cup C$, $T\cap C=\emptyset$, is called a *Halin* graph if $G[T]$ is a tree having no degree-two vertices and $G[C]$ is a cycle whose vertices are the degree-one vertices of $G[T]$. The graph in Fig. 3 is a Halin graph. Clearly, wheels are Halin graphs. (If G is a wheel, let T be the set of spokes of G, and let C be the set of rim edges.) It is natural therefore to ask whether it is possible to determine in polynomial time whether a given matroid is the bicircular matroid of a Halin graph. In this section, it will be shown that this decision problem is NP-hard. The proof employs a construction used by Chandru et al. [2] to show that it is NP-hard to determine whether a matroid is the bicircular matroid of a graph. Before stating the main results of this section, some background material is needed.

Let S be a finite set of integers greater than 1, and let ω denote the product of the integers in S. In addition, suppose Ω is an integer such that $\omega\geq\Omega\geq 2$. The *subset-product problem*, abbreviated *s-p problem*, for (S,Ω) is to determine whether there exists a subset of the integers in S whose product is exactly Ω. The pair (S,Ω)

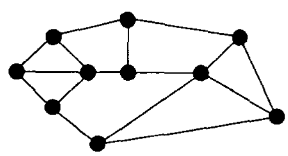

Fig. 3.

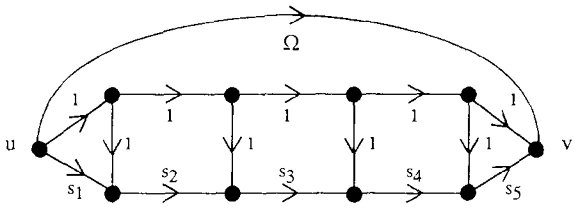

Fig. 4.

is called an *instance* of the s-p problem. The s-p problem is known to be NP-hard [5]. If $|S| \geq 4$, it will be shown that the s-p problem for (S, Ω) can be reduced to the problem of deciding whether a matroid is the bicircular matroid of a Halin graph.

Let (S, Ω) be an instance of the s-p problem where $S := \{s_1, s_2, s_3, s_4, s_5\}$. The weighted digraph in Fig. 4 is an example of an *s-p digraph* for (S, Ω).

It should be clear how to construct s-p digraphs in general for $|S| \geq 4$. The construction given by Chandru et al. is essentially the same as the one above except that in their formulation there is (for purely technical reasons) an extra edge directed from u to v with weight $\omega + 1$. Note that the underlying graph of an s-p digraph is a Halin graph. The proof of the following lemma is identical to the one provided by Chandru et al. if the extra edge described above is ignored.

Lemma 6.1. *If (S, Ω) is an instance of the s-p problem with $|S| \geq 4$ and $(\vec{G}, w)$ is an s-p digraph for (S, Ω), then $(\vec{G}, w)$ has a unit-gain cycle if and only if there exists a subset of the integers in S whose product is exactly Ω.*

A strengthened version of the following lemma appears in Coullard et al. [3].

Lemma 6.2. *Let (S, Ω) be an instance of the s-p problem with $|S| \geq 4$, and let $(\vec{G}, w)$ be an s-p digraph for (S, Ω). Let e be the edge of G with $w(e) = \Omega$, and let u and v denote the ends of e. In addition, let e_1 and e_2 and f_1 and f_2 denote the edges of $H := G \setminus \{e\}$ that have ends u and v respectively. If H' is a connected graph such that $\boldsymbol{B}(H') = \boldsymbol{B}(H)$, then either $H' = H$ or H' can be obtained from H by interchanging e_1 with e_2, f_1 with f_2, or both.*

Lemma 6.3. *If (S, Ω) is an instance of the s-p problem with $|S| \geq 4$ and $(\vec{G}, w)$ is an s-p digraph for (S, Ω),* then $\mathbf{GN}(\vec{G}, w)$ *is the bicircular matroid of a Halin graph if and only if $(\vec{G}, w)$ has no unit-gain cycles.*

Proof. ($\Rightarrow$) Let u, v, e, e_1, f_1, and f_2 be as in the statement of Lemma 6.2, and suppose that $\mathbf{GN}(\vec{G}, w) = \boldsymbol{B}(H)$. If $(\vec{G}', w')$ is the weighted digraph obtained from $(\vec{G}, w)$ by deleting the edge e, then $\mathbf{GN}(\vec{G}', w) = \boldsymbol{B}(H')$ where $H' := H \setminus \{e\}$. (The weighting w' is obtained from w by just restricting w to $E(G) - \{e\}$.) Since $(\vec{G}', w')$ has no unit-gain cycles, $\mathbf{GN}(\vec{G}', w') = \boldsymbol{B}(G')$. Therefore, $\boldsymbol{B}(G') = \boldsymbol{B}(H')$. By Lemma

6.2, H' can be obtained from G' by interchanging e_1 with e_2, f_1 with f_2, or both. Let u' and v' denote the vertices of H' that correspond to the vertices u and v respectively of G'. Since $(\vec{G}', w')$ has no unit-gain cycles, $\{e, e_1, e_2\}$ is a cocircuit of $\mathbf{GN}(\vec{G}, w)$. (A cocircuit of $\mathbf{GN}(\vec{G}, w)$ is a minimal set of edges of G whose deletion increases the number of components every cycle of which is a unit-gain cycle.) Suppose that e does not have u' as an end in H. In this case, $\{e_1, e_2\}$ is a cocircuit of $\boldsymbol{B}(H)$, a contradiction, since $\mathbf{GN}(\vec{G}, w) = \boldsymbol{B}(H)$. Therefore, e has ends u' and v' in H. In summary, H can be obtained from G by interchanging e_1 with e_2, f_1 with f_2, or both.

Now suppose that $(\vec{G}, w)$ has a unit-gain cycle U. Since $\mathbf{GN}(\vec{G}, w) = \boldsymbol{B}(H)$, $K := H[E(U)]$ is a bicycle of H. Since $(\vec{G}', w')$ has no unit-gain cycles, e is an edge of U. Therefore, exactly one of e_1 and e_2 is also an edge of U. Suppose e_1 is an edge of U. Let x be the end of e_1 that is not u, and let x' be the corresponding vertex of H. If H is obtained from G by interchanging e_1 with e_2, then x' is a degree-one vertex of K, a contradiction. Therefore, $G = H$, a contradiction. Consequently, $(\vec{G}, w)$ has no unit-gain cycles.

($\Leftarrow$) If $(\vec{G}, w)$ has no unit-gain cycles, then $\mathbf{GN}(\vec{G}, w) = \boldsymbol{B}(G)$. $\square$

Recall from Section 2 that if N is a gnf matrix, then $M(N) = \mathbf{GN}(\vec{G}, w)$ where $(\vec{G}, w)$ is any associated weighted digraph of N. By combining Lemmas 6.1 and 6.2, the main result of this section is obtained.

Theorem 6.4. *If N is a gnf matrix, the problem of deciding whether $M(N)$ is the bicircular matroid of a Halin graph is* NP-*hard.*

The definition of an s-p digraph can be modified to show that the problem of determining whether a matroid is the bicircular matroid of an outerplanar graph is NP-hard. Like Halin graphs, outerplanar graphs have a simple structure.

If (S, Ω) is an instance of the s-p problem where $S := \{s_1, s_2, s_3, s_4, s_5\}$, the underlying graph of the weighted digraph in Fig. 5 is outerplanar. It should be clear how to construct weighted digraphs like the one in Fig. 5 when $|S| > 5$. Results analogous to those in Lemmas 6.1 and 6.2 can be proved for s-p digraphs structured like the one in Fig. 5.

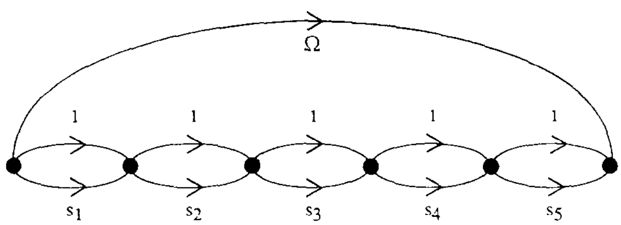

Fig. 5.

References

[1] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (North-Holland, Amsterdam, 1976).
[2] V. Chandru, C.R. Coullard and D.K. Wagner, On the complexity of recognizing a class of generalized networks, Oper. Lett. 4 (1985) 75–78.
[3] C.R. Coullard, J.G. del Greco and D.K. Wagner, Representations of bicircular matroids, Discrete Appl. Math. 32 (1991) 223–240.
[4] C.R. Coullard, J.G. del Greco and D.K. Wagner, Uncovering generalized network structure in matrices, Discrete Appl. Math., to appear.
[5] M.R. Garey and D.S. Johnson, Computers and Intractability (Freeman, San Francisco, CA, 1979).
[6] A.V. Goldberg, S.A. Plotkin and É. Tardos, Combinatorial algorithms for the generalized circulation problem, Math. Oper. Res. 16 (1991) 351–381.
[7] J.E. Hocroft and R.E. Tarjan, Efficient algorithms for graph manipulation, Comm. ACM 16 (1973) 372–378.
[8] J.L. Kennington and R.V. Helgason, Algorithms for Network Programming (Wiley, New York, 1980).
[9] L. Lovász, On graphs not containing independent circuits, Math. Fiz. Lapok 16 (1965) 289–299 (in Hungarian).
[10] L.R. Matthews, Bicircular matroids, Quart, J. Math. Oxford Ser. (2) 28 (1977) 213–228.
[11] L.R. Matthews, Matroids from directed graphs, Discrete Math. 24 (1977) 47–61.
[12] L.R. Matthews and J.G. Oxley, Infinite graphs and bicircular matroids, Discrete Math. 19 (1977) 61–65.
[13] P.D. Seymour, Recognizing graphic matroids, Combinatorica 1 (1981) 75–78.
[14] R. Shull, J.B. Orlin, A. Shuchat and M.L. Gardner, The structure of bases in bicircular matroids, Discrete Appl. Math. 23 (1986) 267–283.
[15] R. Shull, J.B. Orlin, A. Schuchat and M.L. Gardner, Arc gains in bicircular generalized networks, Sloan School of Management, MIT, Cambridge, MA (1987).
[16] R. Shull, A. Shuchat, J.B. Orlin and M.L. Gardner, Recognizing hidden bicircular generalized networks, Sloan School of Management, MIT, Cambridge, MA (1987).
[17] J.M.S. Simões-Pereira, On subgraphs as matroid cells, Math. Z. 127 (1972) 315–322.
[18] J.M.S. Simões-Pereira, On subgraphs of graphs with connected subgraphs as circuits II, Discrete Math. 12 (1975) 55–78.
[19] W.T. Tutte, Introduction to the Theory of Matroids (American Elsevier, New York, 1971).
[20] D.K. Wagner, Connectivity in bicircular matroids, J. Combin. Theory Ser. B 39 (1985) 308–324.
[21] D.J.A. Welsh, Matroid Theory (Academic Press, London, 1976).
[22] T. Zaslavsky, Voltage-graphic matroids, in: A. Barlotti, ed., Matroid Theory and its Applications, Proceedings of CMIE, Varenna, Italy (1980) (Liguore Editore, Naples, 1982) 417–423.
[23] T. Zaslavsky, Bicircular geometry and the lattice of forests of a graph, Quart. J. Math. Oxford Ser. (2) 33 (1982) 493–511.
[24] T. Zaslavsky, Signed graphs, Discrete Appl. Math. 4 (1982) 47–74.